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Slow energy transfer between regions supporting topographic waves

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In a recent paper (Jansons & Johnson 1988) the authors discuss topographic Rossby waves over a random array of seamounts. It is noted that resonance is possible between a hill and an equal and opposite dale but such resonances are mentioned only briefly due to the small likelihood of correctly matched topography in the ocean. The present paper considers the resonances in detail showing how the normal modes formed by frequency splitting at resonance can be combined to give modes that slowly transfer energy from one region supporting topographic waves, across a region where such waves are evanescent, to another region supporting waves. In addition to the simplest case of a hill–dale pair for which an exact energy-transferring mode is obtained, transferring modes are given for a three-hill system, for two hills near a coastal boundary, and for two-basin lakes. The analysis is simplified and the results generalized by extensive use of the invariance of the governing equation under conformal mappings. A transferring mode is given for a dale in a random array of seamounts showing energy leaking slowly from the dale to large distances and returning, with the rate of leakage depending on the area fraction of seamounts. It is concluded that although resonances and transferring modes are not likely to be important in random arrays on infinite planes, they are relevant to numerical models, which are necessarily restricted to finite domains, to coastal seamount chains, and to multi-basin lakes.

1. Introduction

In a recent paper (Jansons & Johnson 1988, hereinafter referred to as I) the authors discuss the topographic Rossby waves over a seamount in the presence of one or more neighbouring seamounts. These waves are the normal modes of oscillation of a homogeneous fluid in the oceanographic limit of slow flow (relative to rapidly rotating axes) over obstacles of small slope and horizontal scale of order the fluid depth or larger, and are governed by the linearized barotropic potential vorticity equation (Rhines 1969). It is shown in I that in a two-mountain system the field seen by one mountain due to the second appears to rotate in the opposite direction to the natural mode of the first and so lies far from resonance. Resonance occurs between a hill and dale (i.e. hill of negative height) of equal and opposite heights. Such exactly matched pairs are extremely unlikely in randomly selected topography on an infinite plane, and so were mentioned only briefly in I. Resonances are, however, more likely in bounded domains where the boundary introduces exact images of the real topography. Thus seamount chains near continental margins and multi-basin lakes can exhibit resonances. Moreover, the necessary restriction of numerical computations to finite domains increases the likelihood of resonance. The frequency of a resonant system is split from the natural frequency of the components in isolation to a higher, '+', and lower '−', frequency, with the magnitude of the splitting varying inversely as the strength of the coupling. A linear combination of the '+' and '−' modes gives a beating mode in which energy is slowly transferred from one component of the system to the other. This paper presents various geometries exhibiting transferring modes, relying extensively on the invariance under conformal mappings of the topographic-wave equation (Johnson 1987*b*) and in particular on invariance under the bilinear or Moebius transformation of inversion with respect to a circle.

Section 2 gives the simplest example – a hill–dale pair. An exact solution is constructed in terms of bipolar coordinates and an approximate solution shows that at large separation (formally, although in practice one radius separation is sufficient) the slow transferring motion has frequency $\omega_1 = \omega_0/d^2$, where d is the separation of centres and ω_0 the frequency of either the hill or dale in isolation. Section 3 discusses resonances in three-mountain systems, giving two samples of transferring modes, one requiring two of the mountains to be of almost equal height and the other for mountains of differing height. Section 4 gives the solution for a dale in a random array of equal and opposite hills. This transferring mode would be absent from an analysis based on the technique reviewed in LeBlond & Mysak (1978) of linearizing the random topography about a non-random mean topography. Section 5 discusses the consequences of these results for numerical experiments, and observations in two-basin lakes.

2. A hill–dale pair

The simplest system exhibiting the slow transfer of energy from one region to another consists of a hill and dale of equal and opposite heights. This section gives a brief derivation of a slow mode using first the far-field approximations of I and then the exact solutions. Note from I that the stream function at $\boldsymbol{x}$ due to a right circular mountain $m_n = (\boldsymbol{x}_n, H_n)$ of height H_n with centre $\boldsymbol{x}_n$, subject to forcing by the incident stream function given by the rotating unit vector

$$\boldsymbol{l} = (\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}}) \exp(\mathrm{i}\omega t),$$

is

$$\nabla\psi(\boldsymbol{x}; m_n; \boldsymbol{l}) = A(\omega/H_n)\, \boldsymbol{l}\cdot\boldsymbol{T}(\boldsymbol{x}-\boldsymbol{x}_n), \tag{2.1}$$

where the amplitude A is a function of ω/H_n alone,

$$A(\omega/H_n) = (2\omega/H_n - 1)^{-1}, \tag{2.2}$$

and

$$\boldsymbol{T}(\boldsymbol{r}) = \begin{cases} 1 & (r \leqslant 1) \\ (1 - 2\hat{\boldsymbol{r}}\hat{\boldsymbol{r}})/r^2 & (r > 1). \end{cases} \tag{2.3}$$

The leading-order approximation to the field due to two seamounts m_0 and m_1 can thus be written

$$\nabla\psi(\boldsymbol{x}; m_0; \boldsymbol{l}) + \nabla\psi(\boldsymbol{x}; m_1; \nabla\psi(\boldsymbol{x}_1; m_0; \boldsymbol{l}). \tag{2.4}$$

For a mountain of height H and dale of depth $-H$ consistency (see I) gives the two roots

$$\omega_\pm = \omega_0(1 \pm d^{-2} + \ldots), \tag{2.5}$$

where d is the separation of centres and $\omega_0 = \frac{1}{2}H$ is the common frequency of either the hill or dale in isolation (ignoring signs). This frequency is split by resonance, with finer splitting at larger separation. Since

$$A(\omega_{\pm}/H) = \pm d^2,$$

the two normal modes are

$$\left.\begin{aligned} \nabla\psi^+ &= \boldsymbol{l}_+ \cdot \boldsymbol{T}(\boldsymbol{x}) + \boldsymbol{l}_+^{\mathrm{R}} \cdot \boldsymbol{T}(\boldsymbol{x}-\boldsymbol{x}_1), \\ \nabla\psi^- &= \boldsymbol{l}_- \cdot \boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{l}_-^{\mathrm{R}} \cdot \boldsymbol{T}(\boldsymbol{x}-\boldsymbol{x}_1), \end{aligned}\right\} \tag{2.6}$$

where $\boldsymbol{l}_{\pm} = (\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}}) \exp(\mathrm{i}\omega t)$

and $\boldsymbol{l}^{\mathrm{R}} = d^2 \boldsymbol{l} \cdot \boldsymbol{T}(\boldsymbol{x}_1) = \boldsymbol{l} \cdot (1 - 2\hat{\boldsymbol{x}}_1 \hat{\boldsymbol{x}}_1)$ is the reflection of $\boldsymbol{l}$ in a line perpendicular to $\hat{\boldsymbol{x}}_1$. Streamlines of the exact solutions for these modes are given in I.

The average of modes (2.6) gives a beating mode, in which the majority of the activity is initially in the neighbourhood of the origin, i.e.

$$\begin{aligned} \tfrac{1}{2}(\nabla\psi^+ + \nabla\psi^-) &= \tfrac{1}{2}(\boldsymbol{l}_+ + \boldsymbol{l}_-) \cdot \boldsymbol{T}(\boldsymbol{x}) + \tfrac{1}{2}(\boldsymbol{l}_+ - \boldsymbol{l}_-)^{\mathrm{R}} \cdot \boldsymbol{T}(\boldsymbol{x}-\boldsymbol{x}_1) \\ &= \boldsymbol{l}_0 \cdot \boldsymbol{T}(\boldsymbol{x}) \cos\omega_1 t + \mathrm{i}\boldsymbol{l}_0 \cdot \boldsymbol{T}(\boldsymbol{x}-\boldsymbol{x}_1) \sin\omega_1 t, \end{aligned} \tag{2.7}$$

where $\boldsymbol{l}_0 = (\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}}) \exp(\mathrm{i}\omega_0 t)$ and $\omega_1 = \omega_0/d^2$. For large separations ω_1 is small and the motion consists of a topographic wave of temporal period $4\pi/H$ initially above the mountain at the origin, transferring its energy to the dale at x_1 and back with period $4\pi d^2/H$. The larger the separation the slower the transfer.

Explicit exact solutions for energy-transferring modes in two-mountain systems follow from the exact solutions of the Appendix to I. Introduce bipolar coordinates

$$x = \frac{\alpha \sinh\xi}{\cosh\xi - \cos\theta}, \quad y = \frac{\alpha \sin\theta}{\cosh\xi - \cos\theta} \tag{2.8}$$

and consider right-circular seamounts of heights H_0 and H_1 bounded by $\xi = \xi_0$ and $\xi = \xi_1$ ($\xi_1 > \xi_0$). The normal modes are given by

$$\psi = F(\xi) \cos(m\theta + \omega t), \tag{2.9}$$

where m is the azimuthal wavenumber (a positive integer),

$$F(\xi) = \begin{cases} \exp[m(\xi-\xi_0)] & (\xi \leqslant \xi_0) \\ \exp[m(\xi-\xi_0)] - (H_0/\omega)\sinh[m(\xi-\xi_0)] & (\xi_0 \leqslant \xi \leqslant \xi_1) \\ \{\exp[m(\xi_1-\xi_0)] - (H_0/\omega)\sinh[m(\xi_1-\xi_0)]\}\exp[m(\xi_1-\xi)] & (\xi \geqslant \xi_1), \end{cases} \tag{2.10}$$

and ω satisfies the dispersion relation

$$(1 - 2\omega/H_0)(1 + 2\omega/H_1) = \beta^{-4m}, \tag{2.11}$$

where $\beta = \exp[\tfrac{1}{2}m(\xi_1 - \xi_0)]$. The two roots are

$$\omega_{\pm} = (H_0 - H_1)/4 \pm \Delta, \tag{2.12}$$

where $\Delta^2 = \tfrac{1}{4}(H_0 + H_1)^2 - H_0 H_1 \beta^{-4m}$. The discriminant Δ measures the closeness of the two frequencies. For a given $H_1 > 0$, a hill–hill pair, the frequencies tend to zero as the hill separation decreases. For $H_1 < 0$, a hill–dale pair, the frequencies become progressively closer as the separation increases (see I). The strongest resonance for a given separation occurs for $H_1 = -H_0 = -H$ (say) when $\omega_{\pm} = \omega_0(1 \pm \beta^{-2m})$, where, as

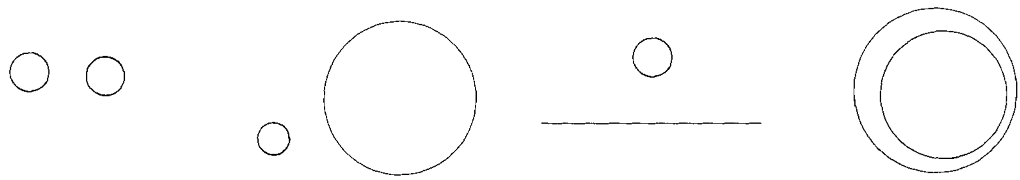

FIGURE 1. Four bathymetries related by inversion in an appropriate circle and so having the same topographic-wave structure. (a) Two circles of equal radius; (b) two circles of differing radius; (c) a circle and rectilinear escarpment; (d) one circle within another. If topographies with these contours are taken to be a hill and dale in (a), (b), and (c) and to be two hills or two dales in (d), then the systems will manifest energy-transferring modes when the step heights are sufficiently close.

above, $\omega_0 = \frac{1}{2}H$ is the frequency of the hill or dale in isolation. Again the average of the modes gives a flow field with slow transfer of energy between the hill and dale. The stream function is

$$\psi = \begin{cases} \sin(m\theta + \omega_0 t)\sin\omega_1 t \exp[m(\xi - \xi_0)] & (\xi \leqslant \xi_0) \\ \{\sin(m\theta+\omega_0 t)\sin\omega_1 t \sinh[m(\xi_1 - \xi)] & \\ \quad + \cos(m\theta + \omega_0 t)\cos\omega_1 t \sinh[m(\xi - \xi_0)]\}/\sinh[m(\xi_1 - \xi_0)] & (\xi_0 \leqslant \xi \leqslant \xi_1) \\ \cos(m\theta + \omega_0 t)\cos\omega_1 t \exp[-m(\xi - \xi_1)], & (\xi \geqslant \xi_1), \end{cases} \tag{2.13}$$

where $\omega_1 = \omega_0/\beta^{2m}$. Resonance can occur for any radii of the topographic features (depending solely on the difference $\xi_1 - \xi_0$) and energy transfer for any azimuthal wavenumber. The higher the mode number and the greater the difference $\xi_1 - \xi_0$, the slower the energy transfer. Figure 1 shows three hill–dale pairs with the same value of $\xi_1 - \xi_0$ and thus related by inversion in an appropriate circle. Figure 1 (a) shows two circles of equal radius, figure 1 (c) a circle and rectilinear escarpment, and figure 1 (b) an intermediate case. Figure 1 (d) is an example of one contour within another, corresponding to a hill with two interior heights. By conformal invariance, all exhibit the same transferring mode.

To compare with the far-field results consider two unit-radius circles with centres d apart. Then $\xi_0 = -\xi_1$ and (see I)

$$\beta = \exp(\xi_0) = \tfrac{1}{2}d + [(\tfrac{1}{2}d)^2 - 1]^{\frac{1}{2}}.$$

At large separations $\beta \approx d$ and so for the dipole mode $\omega_1 \approx \omega_0 d^{-2}$, in agreement with (2.7) above. Figure 2 shows streamlines for $d = 8.125$ for which $\omega_0 = \frac{1}{2}$ and $\omega_1 = \frac{1}{128}$. The patterns are for $t = 0, 0.5\pi, \pi, 32\pi, 32.5\pi, 33\pi, 64\pi, 64.5\pi$ and 65π, corresponding to three consecutive one-eighth-period intervals of the rapid frequency at three consecutive one-eighth-period intervals of the slow, transfer frequency. Figure 2(a) shows the field rotating anticlockwise above the dale, figure 2(b) shows it evenly distributed between hill and dale, and figure 2(c), rotating clockwise above the hill.

Figure 3 shows streamlines for a hill of interior heights unity and two. The outer radius is unity, the inner $\frac{2}{25}\sqrt{6} \approx 0.19$, and the centres are offset by $\frac{2}{25}\sqrt{6}$. This corresponds to $\beta = 2\sqrt{6}$ and $d = \frac{25}{12}\sqrt{6}$, so $\omega_0 = \frac{1}{2}$ and $\omega_1 = \frac{1}{48}$. As in figure 2, the

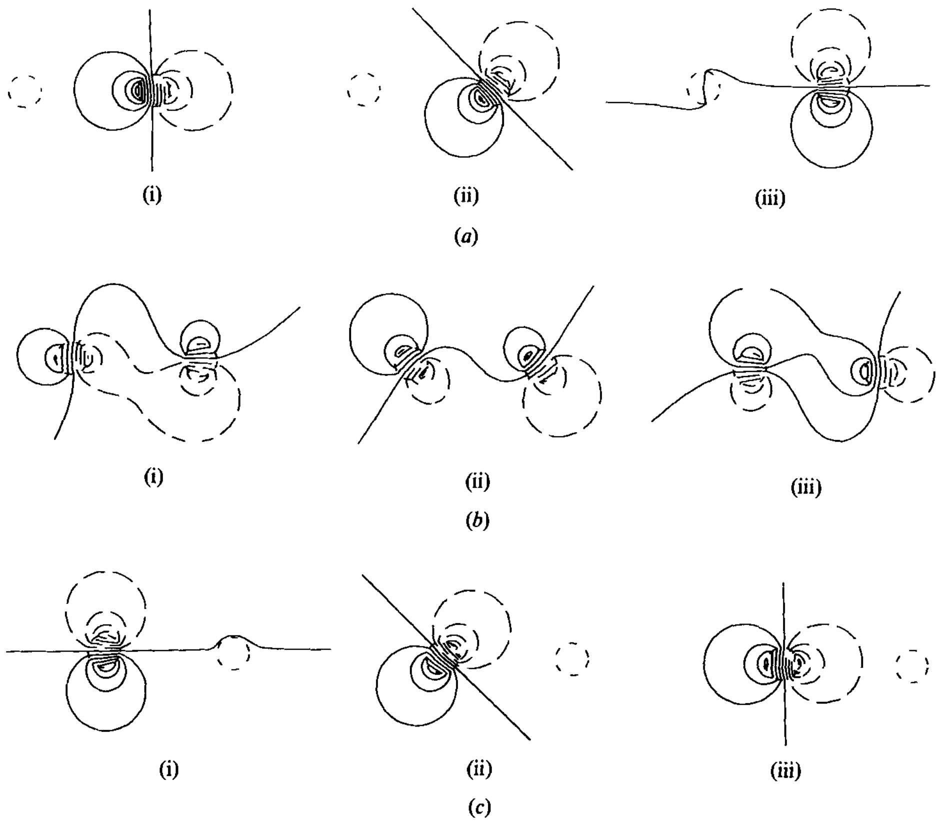

FIGURE 2. Streamlines for a hill and dale, both of unit height and radius with centres a distance 8.125 apart. The period of oscillation of either in isolation is 4π. The period of the slow transferring motion is 256π. The plots are at three consecutive one-eighth-period intervals of the rapid frequency at three consecutive one-eighth-period intervals of the slow frequency, i.e. at times (*a*) (i) 0, (ii) 0.5π, (iii) π, with the disturbance rotating anticlockwise above the dale; (*b*) (i) 32π, (ii) 32.5π, (iii) 33π, evenly distributed between hill and dale; and (*c*) (i) 64π, (ii) 64.5π, (iii) 65π, rotating clockwise above the hill.

patterns are at three consecutive one-eighth-period intervals of the rapid frequency at three consecutive one-eighth-period intervals of the transfer frequency. For the present geometry these are the times $t = 0$, 0.5π, π, 12π, 12.5π, 13π, 24π, 24.5π and 25π. Figure 3(*a*) shows the field rotating anticlockwise, concentrated over the outer contour, figure 3(*b*) shows the field distributed among the contours, and figure 3(*c*) shows it rotating anticlockwise above the interior contour. Transferring modes for two height hills can, of course, be demonstrated most simply for concentric circular contours using polar coordinates.

Although the heights of the hill and dale in these examples are precisely matched, the form of ω_1 shows the strength of the resonance will be unaffected if H_1 differs from $-H_0$ by order d^{-2}. Moreover, the analysis of Johnson (1987*a*) extends straightforwardly to the present geometry, giving solutions for finite-amplitude continuous topography. At large separations these solutions appear locally like the modes over smooth axisymmetric hills (e.g. Rhines 1969), with an additional wavenumber giving an oscillating radial structure to the solution above the sloping sides of the hill and dale.

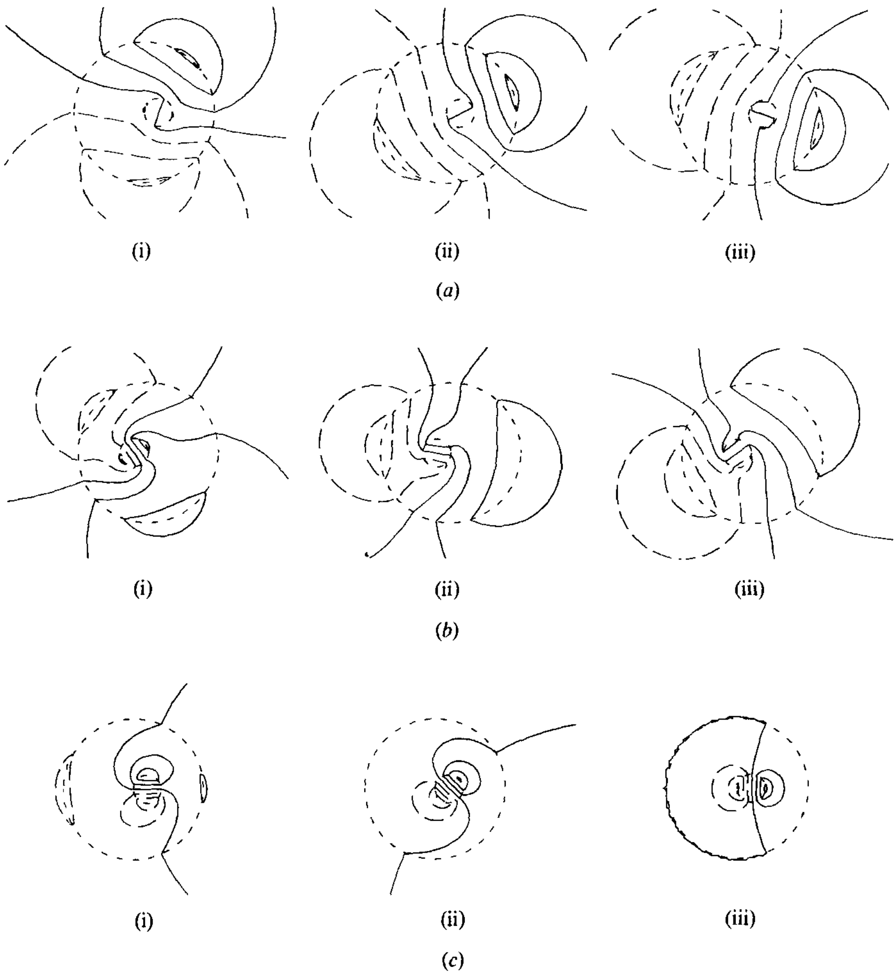

FIGURE 3. As for figure 2 but for a hill of interior heights unity and two. The outer radius is unity, the inner 0.19, and the centres are offset by 0.19. The period of oscillation over either escarpment is 4π. The period of the slow transferring motion is 96π. The plots are at three consecutive one-eighth-period intervals of the rapid frequency at three consecutive one-eighth-period intervals of the slow frequency, i.e. at times (*a*) (i) 0, (ii) 0.5π, (iii) π, with the disturbance above the outer contour; (*b*) (i) 12π, (ii) 12.5π, (iii) 13π, evenly distributed between the contours; and (*c*) (i) 24π, (ii) 24.5π, (iii) 25π, above the inner contour.

3. Three-hill systems

Normal modes of form (2.4) and (2.10) for a pair of identical hills have the same frequency, with each field concentrated over one hill causing a counter-rotating flow over the other. They are degenerate: any linear combination of the two being another normal mode of the same frequency (figure 6*a*). They are also far from resonance and so a system of hills without dales needs at least three hills to exhibit energy-transferring modes. To show that three are sufficient consider three arbitrarily spaced non-intersecting hills, two of the same height and the third different. As the modes of oscillation above the third are not resonant with those of the pair, the qualitative behaviour is unchanged if the third hill is replaced by an island. To justify

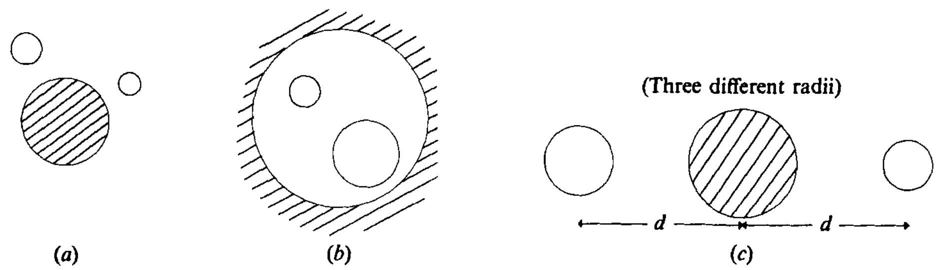

FIGURE 4. Three equivalent geometries related by inversion. (a) Two hills and an island; (b) a two-basin lake; (c) the canonical form, chosen so that the hill centres are each a distance d from the island centre. Any three non-intersecting circles can be conformally mapped so that their centres are collinear although their radii will in general alter.

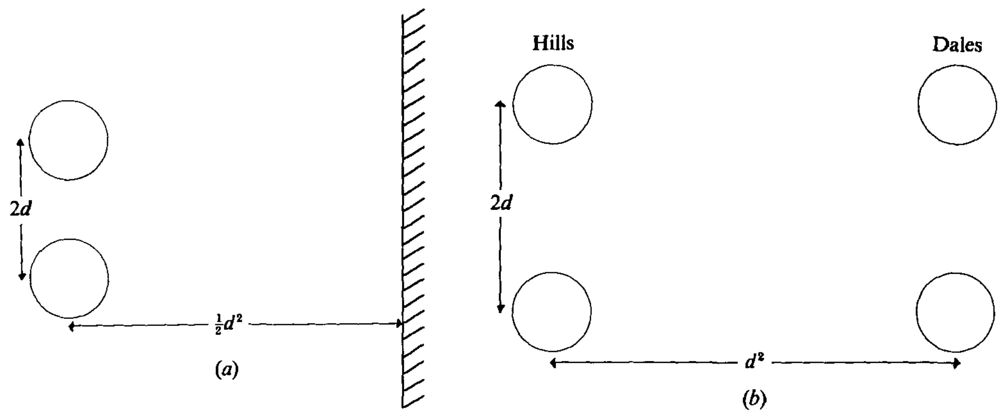

FIGURE 5. (a) The canonical system is mapped conformally to a pair of unit-radii hills a distance $2d$ apart and each approximately a distance $\frac{1}{2}d^2$ from a rectilinear bounding wall by an appropriate inversion. (b) The modes of this system are most conveniently discussed by introducing the images of the hills in the bounding wall.

this replacement note that the modification to the frequency above a hill of height H_0 due to the presence of a hill of height H_1 is given in I, under the far-field approximation, as

$$\omega = \tfrac{1}{2}H_0[1-(1+H_0/H_1)^{-1}d^{-4}+\ldots], \tag{3.1}$$

with d the separation, as before. The factor multiplying d^{-4} varies from $\frac{1}{2}$ to unity as H_1 increases from H_0 to infinity; the expression is otherwise unchanged. Similar changes are expected with more realistic topographies. The system of three circles can be mapped to a canonical system where the centres are collinear and the hill centres are equidistant from the island. This follows by noting that there is a unique circle orthogonal to any three non-intersecting circles and that inversion with respect to a circle centred on this orthogonal circle preserves circles and orthogonality, thus placing the centres of the three original circles on a straight line (figure 4). Such systems are in general non-resonant. Two cases in which the system is resonant and has transferring modes are considered below.

Consider first the case where the two hills have equal radii in the canonical system. Take unit-radius hills with centres a distance d either side of the island centre. By inverting with respect to the circle orthogonal to the hills with centre on the island boundary equidistant from the hills, the system is mapped into one consisting of two

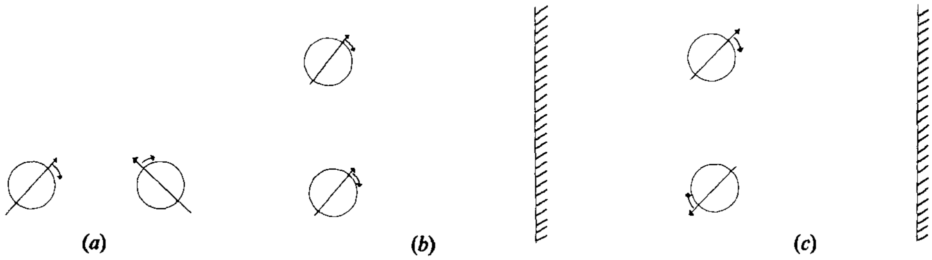

FIGURE 6. A schematic representation of the modes above a hill–hill pair. The directions of the dipoles, both of which rotate clockwise with angular velocity close to ω_0 (see text), are represented by thickened arrows. (a) For an isolated pair the relative orientation of the dipoles is arbitrary. When an island is present, mapped here to an infinite rectilinear boundary, this degeneracy is removed. (b) The '+' mode with dipoles parallel, (c) the '−' mode with dipoles antiparallel.

unit-radius hills on a semi-infinite plane. To leading order in the small parameter d^{-1} the hills are separated by a distance $2d$ and are both a distance $\frac{1}{2}d^2$ from the boundary of the plane (figure 5a). The presence of the island or wall removes the previous degeneracy of the hill–hill pair. The corotating dipoles can no longer differ by an arbitrary phase and two distinct eigenmodes appear – a parallel or '+' mode in which the dipoles are in phase (figure 6b) and an antiparallel or '−' mode in which the dipoles are out of phase by π (figure 6c).

The solid boundary is accommodated by introducing the images in the boundary of the hills. This gives the two-hill–two-dale system of figure 5(b). The frequencies follow from a modified form of the argument leading to (33) of I. The forcing over the image dale corresponding to a forcing $\boldsymbol{l}$ over a hill is the reflection $\boldsymbol{l}^{\mathrm{R}}$ from (2.6), where in this case $\boldsymbol{x}_1$ is the displacement of the dale from the hill. This disturbance $\boldsymbol{l}^{\mathrm{R}}$ in turn makes a contribution at the hill of

$$A(\omega/h)\,\boldsymbol{l}^{\mathrm{R}}\cdot\boldsymbol{T}(-\boldsymbol{x}_1) = A(\omega/H)\,\boldsymbol{l}d^{-4}. \tag{3.2}$$

In the '+' mode the other dale makes the same contribution to leading order in d^{-1}, as to this order the displacements of both dales from a given hill are the same. As in I, the second hill gives a contribution

$$A(\omega/H)\,A(-\omega/H)\,(2d)^{-4}\boldsymbol{l}. \tag{3.3}$$

Thus the consistency condition for the '+' mode is

$$2A(\omega_+/H)\,d^{-4}+A(\omega_+/H)\,A(-\omega_+/H)\,(2d)^{-4} = 1,$$

i.e.
$$1-2\omega_+/H = [2-(1+2\omega_+/H)^{-1}/16]\,d^{-4}. \tag{3.4}$$

Substituting the leading-order solution $\omega_0 = \frac{1}{2}H$ in the right-hand side gives the first correction:

$$\omega_+ = \omega_0(1+\tfrac{63}{32}d^4+\ldots). \tag{3.5}$$

In the '−' mode the image contributions cancel to leading order leaving only contribution (3.3). Thus

$$1-2\omega_-/H = -(1+2\omega_-/H)^{-1}/16d^4, \tag{3.6}$$

giving the first correction:

$$\omega_- = \omega_0(1-\tfrac{1}{32}d^4+\ldots). \tag{3.7}$$

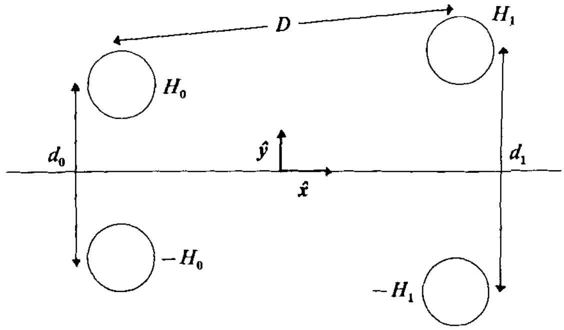

FIGURE 7. Two unit-radius hills near a solid rectilinear wall. If system-0 consisting of mountain H_0 and its image dale has the same frequency in isolation as system-1 consisting of mountain H_1 and its image dale, then the coupled system shows resonant splitting of this common frequency and has modes slowly transferring energy between system-0 to system-1. The basic frequency is $\omega_0 = \frac{1}{2}H_0(1+d_0^{-2}) = \frac{1}{2}H_1(1+d_1^{-2})$ and the transferring frequency is $\omega_1 = \frac{1}{2}(H_1 H_2)^{\frac{1}{2}}/D^2$. This mode does not require the hills to be of the same height.

Unlike the hill–dale system the frequency splitting by resonance in the three-hill system is not symmetric about ω_0.

As in the previous examples, the average of these modes gives a transferring field initially concentrated above one hill but slowly moving to the other. The frequency of the oscillation above a given mountain is increased from ω_0 to

$$\tfrac{1}{2}(\omega_+ + \omega_-) = \omega_0(1 + \tfrac{31}{32}d^4 + \dots). \tag{3.8}$$

The frequency of energy transfer, i.e. the beat frequency, is

$$\tfrac{1}{2}(\omega_+ - \omega_-) = \omega_0/d^4 + \dots, \tag{3.9}$$

decreasing far more rapidly with increasing separation than in the hill–dale example of the previous section.

The example above demonstrates the simplest three-hill resonance. Resonance is not, however, restricted to systems with hills of the same height. Consider the system in figure 7 consisting of two hills $m_0(\boldsymbol{x}_0, H_0)$ and $m_1(\boldsymbol{x}_1, H_1)$, at distances $\frac{1}{2}d_0$ and $\frac{1}{2}d_1$ respectively from a rectilinear solid boundary, with centres separated by a distance D, where d_0 and d_1 are of order $d \gg 1$ and $d \ll D \ll d^2$. If system-0 and system-1 have the same frequency in isolation then the coupled system is resonant. This coupled system is, of course, related to the canonical three-mountain system by inversion with respect to a suitably chosen circle. Introduce the notation

$$\boldsymbol{l}^x = \boldsymbol{l}_0\cdot(1-2\hat{x}\hat{x}), \quad \boldsymbol{l}^y = \boldsymbol{l}_0\cdot(1-2\hat{y}\hat{y}) \tag{3.10}$$

for the reflections of $\boldsymbol{l}$ in lines perpendicular respectively to the x- and y-directions and denote the image dales by m_0^y and m_1^y. Figure 8 shows diagrammatically, to leading order in d/D, the interaction between system-0 and system-1. The forcing above m_0^y corresponding to $\boldsymbol{l}$ above m_0 is the reflection $\boldsymbol{l}^y$. This dale forces most strongly the mountain m_1 giving a forcing in the direction $\boldsymbol{l}^{yx} = -\boldsymbol{l}$ over m_1. The corresponding forcing over the dale m_1^y is in the direction $-\boldsymbol{l}^y$. This forces most strongly the mountain m_0, giving a forcing there with direction $-\boldsymbol{l}^{yx} = \boldsymbol{l}$, as required. It can be shown using the exact directions for the reflections that there is no phase change to any order in d/D introduced in passing round the interaction diagram. To

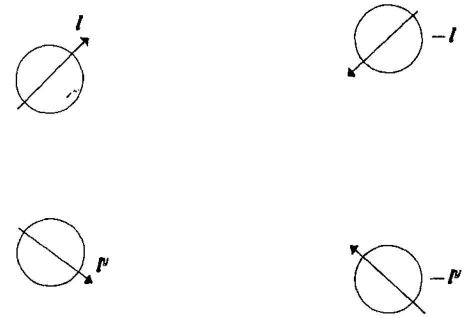

FIGURE 8. A schematic representation of the directions of the dipoles in the resonant mode of the system described in figure 7.

obtain the flow field explicitly, let the forcing at m_0 be $\boldsymbol{l}_0$ and at m_1 be $\boldsymbol{l}_1$. Then the resonant interaction is given by

$$\left.\begin{aligned}\boldsymbol{l}_0 &= \nabla\psi(\boldsymbol{x}_0; m_0^y; \boldsymbol{l}_0^y) + \nabla\psi(\boldsymbol{x}_0; m_1^y; \boldsymbol{l}_1^y),\\ \boldsymbol{l}_1 &= \nabla\psi(\boldsymbol{x}_1; m_0^y; \boldsymbol{l}_0^y) + \nabla\psi(\boldsymbol{x}_1; m_1^y; \boldsymbol{l}_1^y).\end{aligned}\right\} \tag{3.11}$$

Substituting for $\nabla\psi$ from (2.1) gives

$$\boldsymbol{l}_0 = \frac{A_0 \boldsymbol{l}_0^{yy}}{d_0^2} + \frac{A_1 \boldsymbol{l}_1^{yx}}{D^2}, \quad \boldsymbol{l}_1 = \frac{A_0 \boldsymbol{l}_1^{yx}}{D^2} + \frac{A_1 \boldsymbol{l}_1^{yy}}{d_1^2}, \tag{3.12}$$

where $A_n = A(\omega/H_n)$. Hence

$$\begin{pmatrix}\boldsymbol{l}_0\\ \boldsymbol{l}_1\end{pmatrix} = \begin{pmatrix} A_0 d_0^{-2} & -A_1 D^{-2}\\ -A_0 D^{-2} & A_1 d_1^{-2}\end{pmatrix}\begin{pmatrix}\boldsymbol{l}_0\\ \boldsymbol{l}_1\end{pmatrix}. \tag{3.13}$$

Non-trivial solutions to this system require

$$(A_0 d_0^{-2} - 1)(A_1 d_1^{-2} - 1) = A_0 A_1 D^{-4}. \tag{3.14}$$

Substituting for A_n from (2.2) gives

$$[\omega - \tfrac{1}{2}H_0(1 + d_0^{-2})][\omega - \tfrac{1}{2}H_1(1 + d_1^{-2})] = \tfrac{1}{4}H_0 H_1 D^{-4}. \tag{3.15}$$

Now system-0 and system-1 have the same frequency ω^* in isolation so

$$\omega^* = \tfrac{1}{2}H_0(1 + d_0^{-2}) = \tfrac{1}{2}H_1(1 + d_1^{-2}), \tag{3.16}$$

and (3.15) becomes

$$(\omega - \omega^*)^2 = \tfrac{1}{4}H_0 H_1 D^{-4},$$

i.e.

$$\omega = \omega^* \pm \tfrac{1}{2}(H_0 H_1)^{\frac{1}{2}} D^{-2}. \tag{3.17}$$

There is resonant splitting of the frequency, symmetric about $\omega = \omega^*$. Substituting (3.17) in (3.13) gives the ratios of the dipole strengths over the two mountains,

$$\boldsymbol{l}_1 = \left(\frac{H_1}{H_0}\right)\left(\frac{d_1}{d_0}\right)^2 \boldsymbol{l}_0 = \pm\frac{d_1}{d_0}\left(\frac{1+d_1^2}{1+d_0^2}\right)^{\frac{1}{2}} \boldsymbol{l}_0. \tag{3.18}$$

As in the first example, the relative orientation of the dipoles over the hills is arbitrary when they are isolated. When the hills interact the orientation is restricted to either a '+' mode, of frequency higher than ω^*, in which the dipoles are parallel or a '−'

mode, of frequency lower than ω^*, in which the dipoles are antiparallel. In the present example the strengths of the dipoles differ if the hills have different heights, the stronger dipole lying over the higher mountain.

The average of the modes gives a transferring mode with rapid frequency $\omega_0 = \omega^*$ and slow, transferring frequency $\omega_1 = \frac{1}{2}(H_0 H_1)^{\frac{1}{2}}/D^2$, depending on the geometric mean of the mountain heights.

The restriction that d_0 and d_1 are large, and so H_0 and H_1 close, can be relaxed by using the exact solution for system-0 and system-1 obtained by solving the forced problem for a hill near a rectilinear boundary in terms of bipolar coordinates. The form of the solution would be unaltered.

4. Resonant systems containing many mountains

The analysis of I can be extended to demonstrate resonance in a system containing many mountains and dales. The simplest example is of a hill surrounded by a haze of equal and opposite dales or, equivalently, a dale surrounded by a haze of equal and opposite hills. Results are presented below for the case of a dale as this is the more likely occurrence owing to the comparative rarity of deep hollows on the ocean floor. The consistency relation for a dale of depth $-H_0$ surrounded by N hills of height H_0 follows from equation (30) of I as

$$\boldsymbol{l} = \sum_{i=1}^{N} A\left(-\frac{\omega}{H_0}\right) A\left(-\frac{\omega}{H_0}\right) \boldsymbol{l}_0 \cdot \boldsymbol{T}(\boldsymbol{x}_i) \cdot \boldsymbol{T}(\boldsymbol{x}_i). \tag{4.1}$$

Substituting from (2.2) for A gives

$$\left(1 + \frac{2\omega}{H_0}\right)^2 = \sum_{i=1}^{N} [r_i^{-4} + O(r_i^{-6})], \tag{4.2}$$

where r_i is the distance of mountain i from the origin. Thus the leading-order correction to the natural frequency of the dale is given by

$$\omega = -\tfrac{1}{2}H_0 \left[1 \pm \left\{\sum_{i=1}^{N} [r_i^4 + O(r_i^{-6})\right\}^{\frac{1}{2}}\right]. \tag{4.3}$$

There is symmetric splitting about the natural frequency $\omega_0 = -\frac{1}{2}H_0$ and energy is transferred between the dale and the surrounding hills with frequency

$$\omega_1 = \omega_0 \left\{\sum_{i=1}^{N} r_i^4\right\}^{-\frac{1}{2}}. \tag{4.4}$$

The ensemble average frequency for an infinite system of mountains follows by replacing N by infinity in (4.4) and taking an ensemble average over all systems having a dale identical to m_0 at the origin. This gives

$$\langle\omega\rangle_0 = -\tfrac{1}{2}H_0 \left\{1 \pm \left\langle \sum_{i=1}^{\infty} [r_i^{-4} + O(r_i^{-6})] \right\rangle_0^{\frac{1}{2}}\right\}, \tag{4.5}$$

with the subscript 0 indicating that all subscript-0 quantities are held constant in the averaging. As in I, the ensemble average of the sum can be written

$$\left\langle \sum_{i=1}^{\infty} r_i^4 \right\rangle_0 = \int r^{-4} P(r\,|\,0)\, 2\pi r\, \mathrm{d}r, \tag{4.6}$$

where $P(r\,|\,0)$ is the probability of finding a mountain at distance r given there is a dale at the origin. Distribution $P(r\,|\,0)$ includes no contribution from m_0 and for simplicity is taken to be isotropic.

Consider the special case where the pair probability density function is given by

$$P(r\,|\,0) = \begin{cases} 0, & r < a \\ n, & r \geqslant a, \end{cases} \tag{4.7}$$

where n is a constant number density and a is supposed to be much larger than the radius of a mountain and much less than the typical distance between mountains. Then

$$\langle \omega \rangle_0 = -\tfrac{1}{2}H_0\{1 \pm c^{\frac{1}{2}}/a + O(c^{\frac{1}{2}}/a^2, c)\}, \tag{4.8}$$

where $c = \pi n$ is the area fraction of mountains. There is symmetric splitting about $\omega_0 = -\frac{1}{2}H_0$ and energy is transferred from dale to the surrounding haze of equal and opposite mountains with frequency

$$\langle \omega_1 \rangle_0 = \omega_0 c^{\frac{1}{2}}/a + \dots. \tag{4.9}$$

This analysis can be continued following I to determine the effective topography for the ensemble-averaged stream function. For the distribution given by (4.7) the effective topography consists of the original dale together with a further upward step at a distance a from the origin. The transferring-mode stream function thus resembles that of figure 3, with, however, the outer contour at $a \gg 1$. The field is initially concentrated far from the origin (figure 3a), then moves to be concentrated over the dale (figure 3c, with direction of rotation reversed) and so on. It should be noted that the method of linearizing about a background topography (as reviewed in LeBlond & Mysak 1978) would fail to obtain this transferring mode.

5. Discussion

It has been shown that slow energy transfer can occur between widely separated regions supporting topographic waves on an otherwise flat plane, a phenomenon akin to the tunnel effect of quantum dynamics in which energy travels slowly across forbidden regions between accessible states. The hill–dale pair of §2 gives the simplest example, relating resonant frequency splitting to slow transferring modes and allowing explicit solution. Section 3 shows that resonance in systems consisting only of hills requires the height and positioning of the hills to satisfy rigorous restrictions. Resonance is rare in arbitrarily chosen systems, and thus is not expected to be an important effect in an infinite ocean (see I). However, in a strongly bounded domain, one in which topography is present near the walls of the domain, the images introduced by the boundary make resonance more likely. This is particularly relevant to numerical experiments on topographic waves. The experiments are necessarily on a bounded domain with either rigid or periodic boundary conditions, both of which introduce the image system of the topography. For symmetrically placed hills of equal heights (so that they have the same frequency in isolation) resonant splitting introduces slow, transferring modes. Such modes have been observed in numerical integrations (Rhines & Bretherton 1973).

Geophysical examples of strongly bounded domains are given by seamount chains near continental margins and by two-basin lakes, like Lake Michigan. The latter are poorly described by existing analytical models of topographic oscillations in lakes (e.g. Johnson 1987*a*), which either treat the two basins separately or ignore the barrier between the basins. The canonical hill–island–hill geometry of §3 is converted to a two-basin lake by inversion with respect to a circle within the island, thus demonstrating the possibility of modes which evolve slowly over time from being concentrated in one basin to being concentrated in the other. The reported depths of Lake Michigan in Saylor, Huang & Reid (1980) and Schwab (1983) show two basins each of depth around 160 m below a basic lake depth of about 100 m. Free oscillations of the basin could thus show a transferring mode.

The solution in §4 for a dale in a random array of equal and opposite hills gives an example for which the techniques reviewed in LeBlond & Mysak (1978) fail to give the transferring mode.

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